Inner Models and Topos Theory

Philipp Hieronymi

Merton College, University of Oxford

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A Topos

Definition

A category $\mathcal{C}$ is a topos, if it has all finite limits, a subobject-classifier and for all objects a power object.

A subobject-classifier

Definition

We say an object $\Omega \in \mathcal{C}$ and a morphism $\mathbf{true} : 1 \to \Omega$ are a subobject-classifier of $\mathcal{C}$, if for all objects $B \in \mathcal{C}$ and for all monomorphism $m : A \hookrightarrow B$ there is exactly one morphism $f : B \to \Omega$, such that

$$\begin{array}{ccc} A & \longrightarrow & 1 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \mathbf{true}} \\ B & \xrightarrow{f} & \Omega \end{array}$$

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Power-objects

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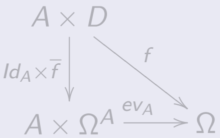
Let A be an object in $\mathcal{C}$. We say that an object $\Omega^A \in \mathcal{C}$ and a morphism $ev_A : A \times \Omega^A \to \Omega$ is a power-object of A, if for every object $D \in \mathcal{C}$ and every morphism $f : A \times D \to \Omega$ there is exactly one morphism $\overline{f} : D \to \Omega^A$, such that



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$$\begin{array}{ccc} A \times D & & \\ {\scriptstyle Id_A \times \overline{f}} \downarrow & \searrow^{f} & \\ A \times \Omega^A & \xrightarrow{ev_A} & \Omega \end{array}$$

commutes.

Examples

Examples of a topos

1. **Set**
2. **Set**$_{\text{fin}}$

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Heyting-algebras in a topos

Given a topos $\mathcal{E}$ and a subobject-classifier Ω in $\mathcal{E}$.

Theorem

Let A be an object in $\mathcal{E}$, then $Hom_{\mathcal{E}}(A, \Omega)$ is a Heyting algebra.

Construction

The greatest element in $Hom_{\mathcal{E}}(A, \Omega)$ is

$$A \xrightarrow{!} 1 \xrightarrow{\textbf{true}} \Omega.$$

The least element is the characteristic morphism of the unique morphism

$$0 \hookrightarrow A.$$

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Let $\wedge : \Omega \times \Omega \to \Omega$ be the characteristic morphism of

$$1 \times 1 \xrightarrow{\mathbf{true} \times \mathbf{true}} \Omega \times \Omega.$$

Let $\vee : \Omega \times \Omega \to \Omega$ be the characteristic morphism of the image of

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Then, for morphisms $f, g \in Hom_{\mathcal{E}}(A, \Omega)$, we define

$$f \wedge g := \wedge \circ (f, g) : A \to \Omega,$$
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Construction

Let $=_\Omega : \Omega \times \Omega \to \Omega$ be the characteristic morphism of the diagonal morphism $\Delta : \Omega \to \Omega \times \Omega$.

Let $\Rightarrow : \Omega \times \Omega \to \Omega$ be the composition

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Quantifiers in a topos

Let A, B be two objects in $\mathcal{E}$. Let $\pi : A \times B \to A$ be the projection in $\mathcal{E}$.

Definition

Define

$$\pi^* : Hom_{\mathcal{E}}(A, \Omega) \to Hom_{\mathcal{E}}(A \times B, \Omega)$$
$$f \mapsto f \circ \pi.$$

Theorem

There are two mappings $\exists_A : Hom_{\mathcal{E}}(A \times B, \Omega) \to Hom_{\mathcal{E}}(A, \Omega)$ and $\forall_A : Hom_{\mathcal{E}}(A \times B, \Omega) \to Hom_{\mathcal{E}}(A, \Omega)$ such that $\exists_A$ is the left-adjoint and $\forall_A$ is the right-adjoint of π^*.

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Structure

Let $\sigma = (S, F, R, K, fct)$ be the signature

1. $S = ob(\mathcal{E})$,
2. $F = \emptyset$,
3. $R = \{\in_A | A \in \mathcal{E}\}$ and $fct(\in_A) := (A, \Omega^A)$,
4. $K = \emptyset$.

Definition

Define a σ-strukture $\mathfrak{E}$ by $\mathfrak{E}_S(A) := A$ and $\mathfrak{E}_R(\in_A) := A \times \Omega^A \xrightarrow{ev_A} \Omega$.

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Interpreting σ-formulae

1. $[\![x_1 \in x_2]\!]_{A,\Omega^A} := A \times \Omega^A \xrightarrow{ev_A} \Omega,$
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Local ZF

Theorem (Extensionality)

$$[\![\forall_{\Omega^A} x \forall_{\Omega^A} y \ x = y \Leftrightarrow (\forall_A z(z \in x \Leftrightarrow z \in y))]\!] = \mathbf{true}.$$

Theorem (Separation)

$$[\![\exists_{\Omega^A} y \forall_A x \quad x \in y \Leftrightarrow \varphi]\!]_{\vec{Z}} = \mathbf{true}_{\vec{Z}}.$$

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From local to global

Let $\mathcal{E}$ be complete and locally small.
Idea: Build a von-Neumann-Hierarchy in $\mathcal{E}$.

Construction

Do this by ordinal induction:

$$V_0 := 0_{\mathcal{E}},$$
$$V_{\alpha+1} := \Omega^{V_\alpha} \text{ and}$$
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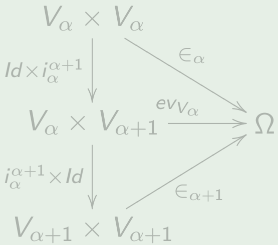
Construction of the embedding

Define embedding $i_\alpha^{\alpha+1} : V_\alpha \hookrightarrow V_{\alpha+1}$ and $\in$-Relation $\in_{\alpha+1}$: $V_{\alpha+1} \times V_{\alpha+1} \to \Omega$ such that



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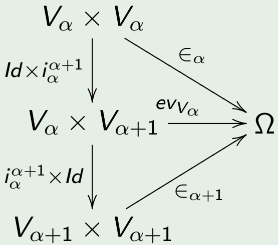

From local to global III

The model

$$[\![x_1 \in x_1]\!]_\alpha := \mathbf{false}_\alpha$$

$$[\![x_1 = x_1]\!]_\alpha := \mathbf{true}_\alpha$$

$$[\![x_1 = x_2]\!]_{\alpha,\beta} :=\, =_{max(\alpha,\beta)} \circ (i_\alpha^{max(\alpha,\beta)} \times i_\beta^{max(\alpha,\beta)})$$

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$$[\![\forall x_1 \ \varphi]\!]_{\vec{\alpha}} := \bigwedge_{\alpha_1 \in On} [\![\forall_{\alpha_1} x_1 \ \varphi]\!]_{\vec{\alpha}}$$

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The model

Definition

Let φ be $\in$-sentence. We say $\mathcal{E} \models \varphi$, if $[\![\varphi]\!] = \textbf{true}$.

Theorem (Fourman)

In this model the rules of intuitionistic predicative calculus are valid.

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$\mathcal{E} \models IZF$ and if $\mathcal{E}$ is boolean, $\mathcal{E} \models ZF$.

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Forcing and Toposes

Let $\mathbb{B}$ a boolean algebra, and $V^{\mathbb{B}}$ be the boolean universe, seen as a category.

Theorem (Higgs)

$V^{\mathbb{B}}$ and $Sh(\mathbb{B})$ are equivalent categories.

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What about Inner Models?

Question: Is there a possibility to construct a subtopos having certain set-theoretical properties?
Idea: Take a subhierarchy of the Von-Neumann-Hierarchy.
Aim: Construct the category of subobjects of this subhierarchy and show that this is a topos.

A subhierarchy

Definition

We say $(m_\alpha : M_\alpha \hookrightarrow V_\alpha)_{\alpha \in On}$ is a subhierarchy, if

(i) for all $\alpha \in On$, $M_\alpha \xrightarrow{m_\alpha} V_\alpha \xrightarrow{i_\alpha^{\alpha+1}} V_{\alpha+1}$ factors through $M_{\alpha+1} \xrightarrow{m_{\alpha+1}} V_{\alpha+1}$.

(ii) for all $\alpha \in On$, $M_{\alpha+1} \xrightarrow{m_{\alpha+1}} V_{\alpha+1}$ factors through $\Omega^{M_\alpha} \xrightarrow{\exists_{m_\alpha}} \Omega^{V_\alpha}$, and

(iii) for all $\lambda \in On$, $lim(\lambda)$, $lim_{\alpha<\lambda} M_\alpha = M_\lambda$.

Subobjects of a subhierarchy

In the following, let $\mathcal{M} := (m_\alpha : M_\alpha \hookrightarrow V_\alpha)_{\alpha \in On}$ be a subhierarchy.

Definition

A subobject $i_A : A \hookrightarrow V_\alpha$ lies in the subhierarchie $\mathcal{M}$, if i_A factors through m_α and its transpose $f_A : 1 \longrightarrow V_{\alpha+1}$ factors through $m_{\alpha+1}$.

The category of subobjects

Definition

Let $i_A : A \hookrightarrow V_\alpha$ and $i_B : B \hookrightarrow V_\beta$ be subobjects. We say $i_A \sim i_B$ iff $i_A = i_\alpha^\beta \circ i_B$.

Definition

We say an equivalence class of $\sim$ lies in $\mathcal{M}$, if the representatives finally lie in $\mathcal{M}$. We call such a equivalence class a subobject of $\mathcal{M}$.

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A nice subhierarchy gives you a topos.

Proposition

Suppose $\mathcal{M}$ is closed under products and Σ_0-formulae, then the subobjects of $\mathcal{M}$ together with morphisms in $\mathcal{M}$ form a category, say $V_{\mathcal{M}}$.

Theorem

Let $\mathcal{M}$ be subhierachy, which is closed under Σ_0-formulae, Products and Exponentiation, and further $Id : 1 \to 1$ and $Id : \Omega \to \Omega$ lie in $\mathcal{M}$, then $V_{\mathcal{M}}$ is a topos.

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Open questions: Completeness? Σ_0-Absoluteness?

Question I: Under which assumptions (to the hierarchy $\mathcal{M}$) is $V_{\mathcal{M}}$ complete? Or, at least, when has $V_{\mathcal{M}}$ enough limits and colimits to construct the von-Neumann-Hierachy in $V_{\mathcal{M}}$?

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